Journal of Engineering Mathematics **25**: 151–163, 1991.

The potential of a Rankine source between parallel planes and in a rectangular cylinder

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Received 12 March 1990; accepted in revised form 26 September 1990

Abstract. When using the boundary element method to compute a potential field, the computations can be substantially reduced if the Green function implicitly satisfies some of the boundary conditions. This savings will be achieved at the expense of having to compute a more complicated Green function. This paper presents efficient formulae for computing the Green functions needed to solve the Laplace equation in domains bounded by a pair of parallel planes or an infinite rectangular cylinder. For both cases, an ascending power series is derived for the region close to the fundamental Rankine source, and a classical eigenfunction expansion is used in the complementary region.

1. Introduction

There are numerous potential problems in which some of the bounding surfaces can be idealized as a pair of infinite parallel planes or an infinite rectangular cylinder. The potential flows about models in towing tanks, water channels and wind tunnels are but a few examples. For all but the simplest model shapes, the velocity potential must be computed numerically. Numerical solutions can be obtained by applying the boundary element method in conjunction with the Rankine source potential $1/|\mathbf{x}-\mathbf{x}'|$, the elementary singular solution of the three-dimensional Laplace equation. Here $\mathbf{x}=(x, y, z)$ is a Cartesian coordinate system and $\mathbf{x}=\mathbf{x}'$ is the location of the source. Using this approach, both the surface of the model and the walls of the tank, channel, or wind tunnel must be discretized into panels. If however the boundary element method is applied in conjunction with a Green function that satisfies the boundary conditions on the walls, only the model surface has to be discretized into panels. Thus far fewer panels will be required to solve the problem numerically and a large savings of computer time will result provided the more complicated Green function can be evaluated rapidly. The present work therfore focuses on efficient methods of evaluating the Green functions for the domains sketched in Fig. 1.

We denote by G_1 the Green function which satisfies periodic boundary conditions on the parallel planes defined by $x=\pm a/2$ and sketched in Fig. 1.1. Of course G_1 is the potential of a Rankine source half way between the parallel planes. The subscript signifies that G_1 is periodic in one direction. We denote by G_2 the potential of a Rankine source located at the center of the rectangular cylinder defined by the two pairs of parallel planes $x=\pm a/2$ and $y=\pm b/2$, and sketched in Fig. 1.2. In this case the subscript indicates that G_2 satisfies periodic boundary conditions in two directions. Greengard [3] discusses the rapid evaluation by a *fast multipole algorithm* of a two-dimensional Green function which is periodic in two directions.

* This work was performed while the author was a postdoctoral researcher in the Ocean Engineering Department of the Massachusetts Institute of Technology.

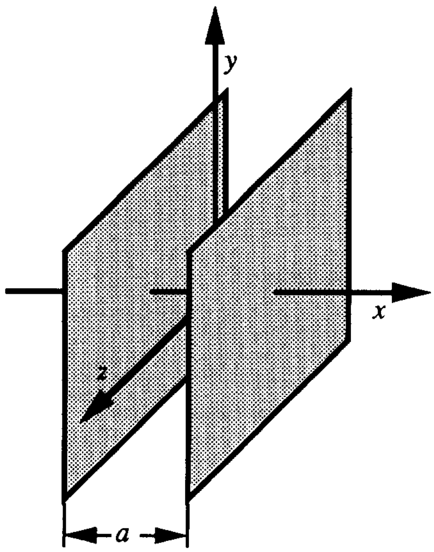

Fig. 1.1. Parallel planes.

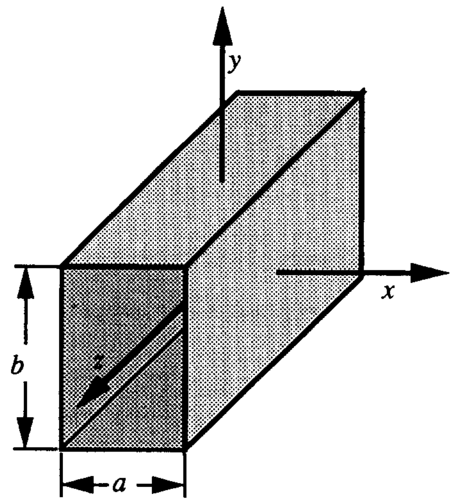

Fig. 1.2. Rectangular cylinder.

Green functions which satisfy homogeneous Neumann or Dirichlet boundary conditions on $x = \pm a/2$ can easily be constructed from G_1, so it is only necessary to consider G_1 in detail. Likewise, Green functions which satisfy homogeneous Neumann or Dirichlet conditions, or combinations of the two, on the boundaries of the rectangular cylinder can be constructed from G_2.

In Section 2 the governing equations satisfied by G_1 and G_2 are formulated and solutions are constructed by the method of images. The resulting infinite series of periodically spaced Rankine sources are not useful for routine computations because they converge very slowly, but they are a starting point for deriving alternative series that converge much more rapidly. Section 3 presents classical eigenfunction expansions of G_1 and G_2 which converge exponentially fast when $R^2 = y^2 + z^2$ is large in the case of G_1, and when z is large in the case of G_2. Ascending power series which converge rapidly in the complementary domain where R and z are small, are derived in Section 4. Finally, Section 5 discusses the numerical implementation of these formulae.

2. Formulation and solution by method of images

The Green functions G_1 and G_2 must satisfy the Poisson equations

$$\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right\} G_1 = -4\pi\delta(y)\delta(z) \sum_{m=-\infty}^{\infty} \delta(x - ma)\,, \tag{1.1}$$

$$\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right\} G_2 = -4\pi\delta(z) \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - ma)\delta(y - nb)\,. \tag{1.2}$$

Solutions of (1.1) and (1.2) may be constructed by superposing infinite sequences of Rankine source singularities:

$$G_1(x, R; a) = (x^2 + R^2)^{-1/2} + \sum_m \{[(x - ma)^2 + R^2]^{-1/2} - |ma|^{-1}\}\,, \tag{2.1}$$

$$G_2(x, y, z; a, b) = (x^2 + y^2 + z^2)^{-1/2} + \sum_{m,n} \{[(x - ma)^2 + (y - nb)^2 + z^2]^{-1/2} - [(ma)^2 + (nb)^2]^{-1/2}\}\,, \tag{2.2}$$

where Σ_m denotes summation over all positive and negative integers m except $m = 0$ and $\Sigma_{m,n}$ denotes summation over all combinations of positive and negative integers m, n except the case $m = n = 0$. A constant is subtracted from each term in the infinite series to make the sequences converge to finite values.

Green functions that satisfy homogeneous Neumann conditions on the parallel planes and rectangular cylinder, denoted here by H superscripts, can be constructed from appropriate combinations of their periodic counterparts:

$$G_1^H(x, R, x'; a) = G_1(x - x', R; 2a) + G_1(x + x', R; 2a)\,, \tag{3.1}$$

$$\begin{aligned} G_2^H(x, y, z, x', y'; a, b) = {} & G_2(x - x', y - y', z; 2a, 2b) + G_2(x + x', y - y', z; 2a, 2b) \\ & + G_2(x + x', y + y', z; 2a, 2b) + G_2(x - x', y + y', z; 2a, 2b)\,. \end{aligned} \tag{3.2}$$

Without loss of generality, we shall henceforth set a equal to unity.

In view of equations (3) and similar relations that can be constructed to obtain Green functions that satisfy homogeneous Dirichlet boundary conditions, it is only necessary to find efficient methods of evaluating G_1 and G_2. The infinite sums in equations (2) converge, but so slowly as to be useless for routine computation. Thus this discussion focuses on the derivation of alternative series expressions that converge more rapidly. Two complementary expressions are needed to compute each of these functions; eigenfunction expansions when R and z are large, and ascending power series when they are small.

3. Eigenfunction expansions

The eigenfunction expansions are classical results and methods of deriving them are presented in such texts as Smythe [6] and Morse and Feshbach [5]. The derivation of the eigenfunction expansion for G_2 is included here for the sake of completeness. The eigenfunction expansions are presented first because of their chronological precedence, and because the eigenfunction expansion of G_1 is used to compute the coefficients of the power series for G_2.

Eigenfunction expansion of G_1

From an identity given in Gradshteyn and Ryzhik [2, equation 8.526] we obtain, after a change of variables, the relation

$$G_1(x, R) = -2\left(\gamma + \ln \frac{R}{2}\right) + \sum_{m=1}^{\infty} K_0(2\pi m R) \cos(2\pi m x)\,, \tag{4}$$

where $K_0(x)$ is the modified Bessel function of zeroth order and γ is Euler's constant. The modified Bessel function decays exponentially for large values of its argument, so the infinite sum in (4) vanishes exponentially fast as $R \to \infty$. This leaves the leading logarithmic term which is what we might have expected; far from the x axis the distribution of discrete sources appears to be continuous and the potential is essentially two-dimensional.

Eigenfunction expansion of G_2

We first define the Fourier transform pair

$$f(x) = \int_{-\infty}^{\infty} e^{i2\pi kx} \tilde{f}(k)\, dk\,, \tag{5.1}$$

$$\tilde{f}(k) = \int_{-\infty}^{\infty} e^{-i2\pi kx} f(x)\, dx\,. \tag{5.2}$$

Taking the Fourier transform of (1.2) with respect to z gives

$$\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - (2\pi k)^2\right\}\tilde{G}_2 = -4\pi \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x-m)\delta(y-nb)\,. \tag{6}$$

Poisson's summation formula can be applied to rewrite the right-hand side of (6) in the form

$$\frac{-4\pi}{b}\left[1 + 2\sum_{m=1}^{\infty} \cos(2\pi mx)\right]\left[1 + 2\sum_{n=1}^{\infty} \cos\left(2\pi n\,\frac{y}{b}\right)\right]. \tag{7}$$

This suggests assuming a solution of the form

$$\tilde{G}_2(x, y, k) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \tilde{\alpha}_{mn}(k) \cos(2\pi mx) \cos\left(2\pi n\,\frac{y}{b}\right), \tag{8}$$

where $\tilde{\alpha}_{mn}$ are as yet unknown coefficients. Substituting (7) and (8) into (6) and collecting coefficients of like terms gives

$$\tilde{\alpha}_{mn} = \frac{\beta_{mn}}{\pi b}\,[m^2 + (n/b)^2 + k^2]^{-1}\,, \quad \text{where } \beta_{mn} = \begin{cases} 1; & m = n = 0\,, \\ 2; & m = 0, n \neq 0 \text{ or } m \neq 0, n = 0\,, \\ 4; & \text{otherwise}\,. \end{cases} \tag{9}$$

Finally we take the inverse Fourier transform of (9) with respect to k to obtain the eigenfunction expansion of G_2:

$$G_2(x, y, z; b) = \lambda - \frac{2\pi}{b}\,|z| + \sum_{m,n} \alpha_{mn} \times \frac{\exp[-2\pi|z|\sqrt{m^2 + (n/b)^2}]}{b\sqrt{m^2 + (n/b)^2}} \cos(2\pi mx) \cos\left(2\pi n\,\frac{y}{b}\right), \tag{10}$$

where λ is a constant, $\Sigma_{m,n}$ again denotes summation over all integers m, n except the combination $m = n = 0$, and $\alpha_{m0} = \alpha_{0n} = 2$, $\alpha_{mn} = 4$. It remains to determine the value of λ

such that (10) is compatible with (2.2). This is accomplished by expressing G_2 in terms of G_1 and then substituting the eigenfunction expansion (4).

G_2 can be viewed as the superposition of an infinite number of G_1 functions aligned in the x-direction. The series (2.1) and (2.2) then give the relation

$$G_2(x, y, z; b) = G_1(x, \sqrt{y^2+z^2}) + \sum_n [G_1(x, \sqrt{(y-nb)^2+z^2}) - G_1(0, |nb|)], \tag{11}$$

where Σ_n denotes summation over all positive and negative integers n except $n = 0$. Once again, a finite constant must be subtracted from each term in the infinite series in order to make it converge. Substituting the eigenfunction expansion (4) for all occurrences of G_1 in (11) except the first gives

$$\begin{aligned} G_2(x, y, z; b) = {} & G_1(x, \sqrt{y^2+z^2}) \\ & - 2\sum_{n=1}^{\infty} [\ln\sqrt{(y+nb)^2+z^2} + \ln\sqrt{(y-nb)^2+z^2} - 2\ln(nb)] \\ & + 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} [K_0(2\pi m\sqrt{(y+nb)^2+z^2})\cos(2\pi mx) - K_0(2\pi mnb)] \\ & + 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} [K_0(2\pi m\sqrt{(y-nb)^2+z^2})\cos(2\pi mx) - K_0(2\pi mnb)]. \end{aligned} \tag{12}$$

It is convenient to define a complex variable $\zeta = y + iz$ and replace the first infinite summation in (12) by the infinite product

$$-2\ln\left|\prod_{n=1}^{\infty}\left\{1 - \left(\frac{\zeta}{nb}\right)^2\right\}\right|. \tag{13}$$

Then using the identity

$$\sin\theta = \theta\prod_{n=1}^{\infty}\left\{1 - \left(\frac{\theta}{n\pi}\right)^2\right\}, \tag{14}$$

equation (12) can be rewritten as

$$\begin{aligned} G_2(x, y, z; b) = {} & G_1(x, \sqrt{y^2+z^2}) - 2\ln\left|\frac{\sin(\pi\zeta/b)}{(\pi\zeta/b)}\right| - 8\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K_0(2\pi mnb) \\ & + 4\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} [K_0(2\pi m\sqrt{(y+nb)^2+z^2}) + K_0(2\pi m\sqrt{(y-nb)^2+z^2})]\cos(2\pi mx). \end{aligned} \tag{15}$$

The first double summation on the right side of (15) is a constant which is evaluated once and for all for a given b value. Equation (15) has uses other than the present one of determining the constant λ; it is also needed to evaluate some of the coefficients in the power series expansion of G_2 and provides a means of confirming both the power series and the eigenfunction expansions. Returning to our original purpose, we substitute (4) for G_1 in (15), set $(x, y) = (0, 0)$, and let z tend to infinity to obtain

$$\lim_{z\to\infty} G_2(0, 0, z; b) = -2\gamma + 4\ln(2) - \frac{2\pi}{b}|z| + 2\ln\left(\frac{\pi}{b}\right) - 8\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K_0(2\pi mnb). \tag{16}$$

Taking the same limit of the eigenfunction expansion (10) and comparing with (16) gives

$$\lambda = 2\Big((\ln \frac{4\pi}{b} - \gamma\Big) - 8 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} K_0(2\pi mnb) . \tag{17}$$

Of course when numerically implementing (10), λ can be evaluated once and for all for a given b value.

4. Ascending power series expansions

The power series are derived by Taylor expanding each of the terms in equations (2) about the points defined by the intersections of the planes of symmetry of the Green functions. This produces series in even powers of the independent variables. The coefficients of the terms in these power series are themselves infinite series which depend only on the integers m and n. These infinite series are known explicitly in the case of G_1, but not for G_2 where some effort is required to compute them. In both cases however, the constants multiplying the infinite series have been found in closed form by inspection.

Power series expansion of G_1

The function G_1 has planes of symmetry $x = 0$ and $x = 1/2$ and furthermore

$$G_1(x, R) = G_1(1 - x, R) ,$$

therefore it is only necessary to evaluate G_1 for the ranges $0 \leqslant x \leqslant 1/2$, $R \geqslant 0$ of its arguments. Two complementary power series of G_1 are derived here by Taylor expanding about the points $(x, R) = (0, 0)$ and $(1/2, 0)$, henceforth to be represented in vector notation by $\mathbf{x}_0$ and $\mathbf{x}_1$ respectively. These points were chosen because they are the intersections of the axis of symmetry and the planes of symmetry, therefore the Taylor series will involve only even powers of x and R. The expansions about the two points are denoted by (0) and (1) superscripts respectively.

The Taylor series of an arbitrary function $f(x, y)$ of two variables that is symmetric with respect to both variables about $(x, y) = (x_s, y_s)$ is given by

$$f(x, y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{(2i + 2j)!} \binom{2i + 2j}{2j} \left[\frac{\partial^{2i+2j} f}{\partial x^{2i}\, \partial y^{2j}} \right]_{\mathbf{x}=\mathbf{x}_s} (x - x_s)^{2i} (y - y_s)^{2j} , \tag{18}$$

where

$$\binom{2i + 2j}{2j} = \frac{(2i + 2j)!}{(2i)!(2j)!}$$

denotes the binomial coefficients. The main task is to evaluate the partial derivatives of f. We begin by rewriting (2.1) in the alternative forms

$$G_1(x, R) = \begin{cases} g_0(x, R) & + H_1^{(0)}(x, R) \\ g_0(x, R) + g_1(x, R) & + H_1^{(1)}(x, R) \end{cases} , \tag{19}$$

where

$$g_m(x, R) = [(m - x)^2 + R^2]^{-1/2}$$

and

$$H_1^{(0)}(x, R) = \sum_m [g_m(x, R) - g_m(0, 0)] ,$$

$$H_1^{(1)}(x, R) = H_1^{(0)}(x, R) - [g_1(x, R) - g_1(0, 0)] .$$

Next we apply the formula (18) to each term of H_1, or in other words to the generic term g_m to obtain the formula

$$\left[\frac{\partial^{2i+2j} g_m}{\partial x^{2i}\, \partial R^{2j}}\right]_{x=x_s} = (-)^j \frac{(2j-1)!!}{(2j)!!} \binom{2i+2j}{2j} [g_m(x_s, 0)]^{-2i-2j-1} , \quad s = 0, 1 , \tag{20}$$

where !! indicates the double factorial. For example,

$$8!! = 8 \cdot 6 \cdot 4 \cdot 2 , \qquad 7!! = 7 \cdot 5 \cdot 3 \cdot 1 ,$$

and by definition

$$0!! = 1 , \qquad (-1)!! = 1 .$$

The i- and j-dependent factors in (18) and (20) are common to all terms of H_1 and can be moved outside the m-summation to obtain

$$H_1^{(s)} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-)^j \binom{2i+2j}{2j} \frac{(2j-1)!!}{(2j)!!}\, c_{ij}^{(s)} (x - x_s)^{2i} R^{2j} , \quad s = 0, 1 , \tag{21}$$

where

$$c_{ij}^{(0)} = \begin{cases} H_1^{(0)}(0, 0) ; & i = j = 0 \\ \sum_m [g_m(0, 0)]^{-2i-2j-1} ; & \text{otherwise} \end{cases}$$

$$c_{ij}^{(1)} = \begin{cases} H_1^{(1)}(\tfrac{1}{2}, 0) ; & i = j = 0 \\ \sum_m [g_m(\tfrac{1}{2}, 0)]^{-2i-2j-1} - [g_1(\tfrac{1}{2}, 0)]^{-2i-2j-1} ; & \text{otherwise} . \end{cases}$$

Here we note from the definitions (19) that

$$H_1^{(0)}(0, 0) = 0 , \tag{22}$$

$$H_1^{(1)}(\tfrac{1}{2}, 0) = 4(\ln 2 - 1) , \tag{23}$$

where the latter result arises from an identity in Gradshteyn and Ryzhik [2, p. 8]. We also observe that

$$\sum_m [g_m(0,0)]^{-2i-2j-1} = 2\sum_{m=1}^{\infty} m^{-2i-2j-1} = 2\zeta(2i+2j+1)\,, \tag{24}$$

$$\sum_{m=-\infty}^{\infty} [g_m(\tfrac{1}{2},0)]^{-2i-2j-1} = 2\sum_{m=1}^{\infty}\left(m-\frac{1}{2}\right)^{-2i-2j-1} = 2(2^{2i+2j+1}-1)\zeta(2i+2j+1)\,, \tag{25}$$

where $\zeta(s)$ is the Riemann-zeta function. The latter expression has been derived from an identity in Abramowitz and Stegun [1]. The summation in (25) is over all m, thus (25) gives the coefficients of a power series representation of G_1 instead of H_1. These coefficients grow geometrically as $i+j$ increases ($\zeta(s) = \mathrm{O}(1)$ for $s \gg 1$), hence the series converges very slowly if at all. Removing the terms corresponding to $m = 0$ and $m = 1$ from (25) gives the coefficients of the desired power series for $H_1(s)$, all of which are O(1).

Summarizing these results, we have

$$c_{ij}^{(0)} = \begin{cases} 0\,; & i=j=0 \\ 2\zeta(2i+2j+1)\,; & \text{otherwise} \end{cases} \tag{26}$$

$$c_{ij}^{(1)} = \begin{cases} 4(\ln 2 - 1) & ;\ i=j=0 \\ 2[(2^{2i+2j+1}-1)\zeta(2i+2j+1) - 2^{2i+2j+1}]\,; & \text{otherwise}\,. \end{cases} \tag{27}$$

This concludes the derivation of the power series representations of G_1. The coefficients have been obtained in closed form and can be evaluated once and for all. Newman [4] has derived a series identical to (21), (26) starting from an integral representation of G_1.

Power series expansion of G_2

The same approach can be followed to derive power series representations of the function G_2. G_2 has planes of symmetry at $x = 0$, $x = 1/2$, $y = 0$, $y = b/2$, and $z = 0$, therefore we are only interested in evaluating it in the region $0 \leqslant x \leqslant 1/2$, $0 \leqslant y \leqslant b/2$, $z \geqslant 0$. Four complementary power series are derived here by Taylor expanding about the four points $\mathbf{x} = (x_s, y_s, 0) = \mathbf{x}_s$, $s = 0, 1, 2, 3$, defined by the intersections of the planes of symmetry. These points and the corresponding power series are numbered as shown in Fig. 2. The power series for G_2, like those for G_1, will involve only even powers of the independent variables.

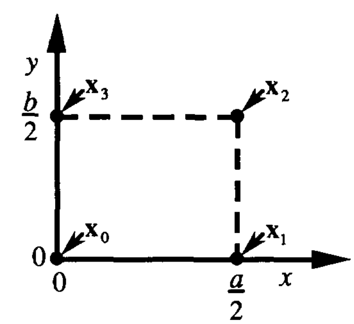

Fig. 2. Scheme for numbering points where Taylor expansions of G_2 are performed.

We begin by rewriting the series (2.2) in the form

$$G_2(x,y,z;b) = g_{00}(x,y,z;b) + \cdots + H_2^{(s)}(x,y,z;b)\,, \quad s = 0,1,2,3, \tag{28}$$

where

$$g_{mn}(x,y,z;b) = [(m-x)^2 + (nb-y)^2 + z^2]^{-1/2}$$

and

$$H_2^{(s)}(x,y,z;b) = \sum_{m,n} [g_{mn}(x,y,z;b) - g_{mn}(0,0,0;b)]\,.$$

The terms represented by $\cdots$ in (28) may be other image-source potentials in the neighborhood of $\mathbf{x}_s$, and $\Sigma_{m,n}$ denotes summation over all m, n except those values corresponding to the $\cdots$ terms. This is a much looser definition of $\Sigma_{m,n}$ than that given in connection with (2.2). For example, if $1/2 < b < 1$, the $\cdots$ terms for the expansion about $\mathbf{x}_0$ would be $g_{0,1}(x, y, z; b)$ and $g_{0,-1}(x, y, z; b)$. Figure 3 indicates which terms are treated separately for $b = 1$ and $b = 1/2$.

The Taylor expansion of H_2 with respect to z alone, about the $z = 0$ plane, is obtained by setting $i = 0$ in (18) and substituting H_2 for f. We also change the j index to k. Once again the differential operators in (18) can be moved inside the summation over m and n, thus the corresponding derivatives of g_{mn} for arbitrary m, n are needed. By analogy with (20),

$$\left[\frac{\partial^{2k} g_{mn}}{\partial z^{2k}}\right]_{z=0} = (-)^k \frac{(2k-1)!!}{(2k)!!} [g_{mn}(x, y, 0; b)]^{-2k-1} . \tag{29}$$

We next obtain the Taylor expansion of $[g_{mn}]^{-2k-1}$ with respect to x and y, about the point $\mathbf{x} = \mathbf{x}_s$ by again substituting in (18). The necessary derivatives were evaluated up to the order $i + j + k < 5$ and the following formula was obtained by inspection:

$$\left[\frac{\partial^{2i+2j} g_{mn}^{-2k-1}}{\partial x^{2i}\, \partial y^{2j}}\right]_{\mathbf{x}=\mathbf{x}_s} = \frac{1}{(2k-1)!!} \sum_{p=0}^{i} \sum_{q=0}^{j} (-)^{p+q} \binom{2i}{2p} \binom{2j}{2q} (2i + 2j + 2k + 2p + 2q + 1)!!$$
$$\times (2i - 2p - 1)!!(2j - 2q - 1)!! m^{2p} (nb)^{2q} [g_{mn}(x_s, y_s, 0; b)]^{2(i+j+k+p+q)+1} . \tag{30}$$

The only factors that depend on m, n are the last three on the right side of (30), so the order of summations can be interchanged to give, finally,

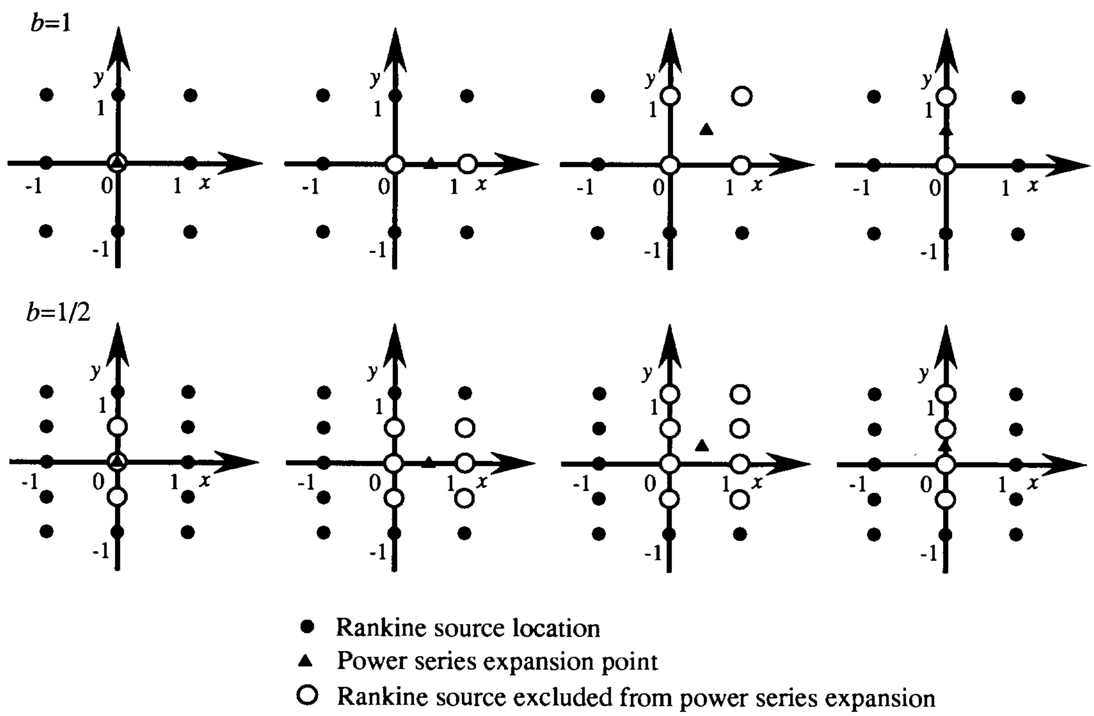

Fig. 3. Rankine sources that are excluded from power-series expansions for $b = 1$ and $b = 1/2$.

$$H_2^{(s)} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-)^{i+j+k}}{(2k)!!(2i+2j)!}\binom{2i+2j}{2j} c_{ijk}^{(s)}(x-x_s)^{2i}(y-y_s)^{2j}z^{2k}\,, \tag{31}$$

where

$$\begin{aligned} c_{ijk}^{(s)} = &\sum_{p=0}^{i}\sum_{q=0}^{j} (-)^{p+q}\binom{2i}{2p}\binom{2j}{2q}(2i+2j+2k+2p+2q+1)!! \\ &\times (2i-2p-1)!!(2j-2q-1)!!\,d_{ijkpq}^{(s)}\,, \end{aligned} \tag{32}$$

$$d_{ijkpq}^{(s)} = \sum_{m,n} (m)^{2p}(nb)^{2q}[g_{mn}(x_s, y_s, 0; b)]^{2(i+j+k+p+q)+1}\,. \tag{33}$$

The coefficients $d_{ijkpq}^{(s)}$ depend only on the aspect ratio b which is fixed for a particular problem, so they can be computed once and for all for a given b value. The remainder of this section discusses ways of evaluating these coefficients.

When $i+j+k>2$ the double sum in (33) converges fast enough to be summed directly, thus our main concern is with the coefficients whose indices satisfy $i+j+k=0$ and $i+j+k=1$. We observe for $i+j+k=0$ that,

$$d_{00000}^{(s)} = H_2^{(s)}(x_s, y_s, 0; b)\,, \quad s = 0, 1, 2, 3\,. \tag{34}$$

The utility of (15) should now be apparent; it can be substituted for the right side of (34) after subtracting the appropriate near-by-image potentials. Even though (15) is not suitable for general-purpose use, it is efficient enough for the one-time evaluation required by (34). The coefficient $d_{00000}^{(0)}$ is trivial of course, as may be seen from (28).

The coefficients corresponding to $i+j+k=1$ are

$$d_{10000}^{(s)} = d_{01000}^{(s)} = d_{00100}^{(s)} = \sum_{m,n} [g_{mn}(x_s, y_s, 0; b)]^3\,, \tag{35}$$

$$d_{10010}^{(s)} = \sum_{m,n} (m-x_s)^2[g_{mn}(x_s, y_s, 0; b)]^5\,, \tag{36}$$

$$d_{01001}^{(s)} = \sum_{m,n} (nb-y_s)^2[g_{mn}(x_s, y_s, 0; b)]^5\,. \tag{37}$$

From the definition (28), we observe that

$$\left[\frac{\partial^2 H_2^{(s)}}{\partial x^2}\right]_{\mathbf{x}=\mathbf{x}_s} = -d_{10000}^{(s)} + 3d_{10010}^{(s)}\,,$$

$$\left[\frac{\partial^2 H_2^{(s)}}{\partial y^2}\right]_{\mathbf{x}=\mathbf{x}_s} = -d_{10000}^{(s)} + 3d_{01001}^{(s)}\,.$$

Adding these expressions and using the definitions (35) through (37) gives

$$d_{10000}^{(s)} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)H_2^{(s)}\bigg|_{\mathbf{x}=\mathbf{x}_s} = -\frac{\partial^2 H_2^{(s)}}{\partial z^2}\bigg|_{\mathbf{x}=\mathbf{x}_s}\,, \tag{38}$$

$$d^{(s)}_{10010} = \frac{1}{3}\left(\frac{\partial^2 H_2^{(s)}}{\partial x^2}\bigg|_{\mathbf{x}=\mathbf{x}_s} + d^{(s)}_{10000}\right), \tag{39}$$

$$d^{(s)}_{01001} = \frac{1}{3}\left(\frac{\partial^2 H_2^{(s)}}{\partial x^2}\bigg|_{\mathbf{x}=\mathbf{x}_s} + d^{(s)}_{10000}\right). \tag{40}$$

The second equality in (38) arises from the Laplace equation but is not very useful. The second derivatives of H_2 can be evaluated from (15) which is easily differentiated twice with respect to x, but not y. However, interchanging the x and y arguments in (15) and replacing b by its reciprocal gives an expression which is easily differentiated twice with respect to y. This amounts to constructing G_2 from an infinite series of G_1 functions that are aligned in the y instead of the x direction. By this process one arrives at the interesting result

$$\sum_{\substack{m,n \\ m=n\neq 0}} [(m)^2 + (nb)^2]^{-3/2} = 4(1+b^{-3})\zeta(3) - 32\pi^2 \times \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^2\left[K_0(2\pi mnb) + b^{-3}K_0\left(\frac{2\pi mn}{b}\right)\right]. \tag{41}$$

Both sides involve infinite double sums, but the one on the right side converges exponentially fast.

5. Numerical evaluation

Two FORTRAN subroutines, named GREEN1 and GREEN2, have been developed to evaluate G_1 and G_2 respectively, and their first derivatives. GREEN1 is completely self contained, but GREEN2 is accompanied by the subroutine G2COEF which evaluates the power-series coefficients. The aspect ratio b is the only input to G2COEF which must be called once for a given b value.

The eigenfunction expansion is used for $R > 1/2$ in the case of G_1, and for $z > 1/4$ in the case of G_2. The GREEN1 subroutine has been checked by comparing the power series and the eigenfunction expansion in an intermediate region, and comparing both with a brute-force evaluation of the series (2.1). The availability of three different ways of computing G_2, not counting the brute-force approach, made checking it even easier. The expression (15) is useful at least in the range $0 < z < 2$, so it has been used to confirm both the power series and the eigenfunction expansions. Also, the power series have been checked against each other in regions where they overlap. To achieve an absolute accuracy of 10^{-6} while minimizing computational effort, both the eigenfunction expansions and the power series are truncated according to the magnitudes of their arguments.

On a VAX 11-750 minicomputer, the average CPU time required for one evaluation of the function and its derivatives was 1.0 ms for G_1 and 2.2 ms for G_2. This compared with an execution time of 0.2 to 0.6 ms for one evaluation of the Bessel function $J_0(x)$, using a subroutine from the International Mathematical and Statistical Library (IMSL). G_1 and G_2 are of course two and three parameter functions respectively while $J_0(x)$ depends on one. Further improvement in the computational times could probably be achieved by converting the power series to economized Chebyshev polynomials.

Both functions are tabulated over selected ranges in Tables 1 and 2.

Table 1. Values of $G_1 - (x^2 + R^2)^{-1/2}$ and its derivatives computed by subroutine GREEN1

x	R	$G_1 - (x^2 + R^2)^{-1/2}$	$\frac{\partial}{\partial x}$	$\frac{\partial}{\partial R}$
0.000	0.0	0.000000	0.000000	0.000000
	0.1	−0.011943	0.000000	−0.237338
	0.2	−0.046877	0.000000	−0.457090
	0.5	−0.260007	0.000000	−0.904676
	1.0	−0.764466	0.000000	−1.024870
	5.0	−3.187013	0.000000	−0.384870
0.125	0.0	0.038078	0.617607	0.000000
	0.1	0.025144	0.601141	−0.256796
	0.2	−0.012553	0.555364	−0.491945
	0.5	−0.238761	0.340854	−0.948442
	1.0	−0.757822	0.105776	−1.040528
	5.0	−3.186950	−0.015351	−0.377578
0.250	0.0	0.158883	1.344544	0.000000
	0.1	0.142479	1.303091	−0.324715
	0.2	0.095213	1.189623	−0.611733
	0.5	−0.174358	0.691902	−1.088349
	1.0	−0.738284	0.205233	−1.086861
	5.0	−3.186763	−0.021044	−0.360086
0.375	0.0	0.385609	2.350501	0.000000
	0.1	0.361245	2.257494	−0.479525
	0.2	0.292549	2.010935	−0.875285
	0.5	−0.065219	1.055924	−1.351002
	1.0	−0.707059	0.291607	−1.161568
	5.0	−3.186453	−0.013253	−0.342795
0.500	0.0	0.772588	3.999971	0.000000
	0.1	0.729617	3.771479	−0.836749
	0.2	0.613138	3.201649	−1.443005
	0.5	0.089450	1.414214	−1.780607
	1.0	−0.666226	0.357771	−1.259716
	5.0	−3.186020	0.003941	−0.335850

Table 2. Values of $G_2 - (x^2 + y^2 + z^2)^{-1/2}$ and its derivatives for $b = 1/2$, computed by subroutine GREEN2

x	y	z	$G_2 - (x^2 + y^2 + z^2)^{-1/2}$	$\frac{\partial}{\partial x}$	$\frac{\partial}{\partial y}$	$\frac{\partial}{\partial z}$
0.00	0.00	0.0	0.000000	0.000000	0.000000	0.000000
		0.1	−0.159338	0.000000	0.000000	−3.135674
		0.2	−0.609190	0.000000	0.000000	−5.745934
		0.5	−3.056511	0.000000	0.000000	−9.801013
		1.0	−8.516259	0.000000	0.000000	−11.613522
		5.0	−57.989231	0.000000	0.000000	−12.526371
0.10	0.00	0.0	−0.027719	−0.503244	0.000000	0.000000
		0.1	−0.183826	−0.443226	0.000000	−3.074917
		0.2	−0.625947	−0.299303	0.000000	−5.659138
		0.5	−3.054331	0.052323	0.000000	−9.772597
		1.0	−8.512728	0.070837	0.000000	−11.619303
		5.0	−57.989189	0.000800	0.000000	−12.526395

Table 2. (continued)

x	y	z	$G_2-(x^2+y^2+z^2)^{-1/2}$	$\frac{\partial}{\partial x}$	$\frac{\partial}{\partial y}$	$\frac{\partial}{\partial z}$
0.20	0.00	0.0	−0.082516	−0.475130	0.000000	0.000000
		0.1	−0.231385	−0.399834	0.000000	−2.938379
		0.2	−0.656146	−0.219544	0.000000	−5.464175
		0.5	−3.042821	0.199806	0.000000	−9.715502
		1.0	−8.502015	0.143868	0.000000	−11.638036
		5.0	−57.989071	0.001596	0.000000	−12.526466
0.50	0.00	0.0	0.284722	4.000000	0.000000	0.000000
		0.1	0.127490	3.771465	0.000000	−3.102662
		0.2	−0.320694	3.201644	0.000000	−5.761146
		0.5	−2.823080	1.414214	0.000000	−10.119673
		1.0	−8.425631	0.357771	0.000000	−11.804031
		5.0	−57.988239	0.003941	0.000000	−12.526963
0.10	0.05	0.0	0.016031	−0.591471	1.762578	0.000000
		0.1	−0.144186	−0.519884	1.594885	−3.151499
		0.2	−0.595782	−0.351189	1.209938	−5.762772
		0.5	−3.046202	0.044833	0.324163	−9.809973
		1.0	−8.511500	0.070474	0.049008	−11.622927
		5.0	−57.989182	0.000799	0.000400	−12.526401
0.20	0.10	0.0	0.050546	−0.966998	2.699587	0.000000
		0.1	−0.109806	−0.831244	2.457633	−3.153374
		0.2	−0.561488	−0.518904	1.897521	−5.762466
		0.5	−3.014643	0.151282	0.556173	−9.836324
		1.0	−8.497340	0.141192	0.092850	−11.651405
		5.0	−57.989029	0.001595	0.000798	−12.526490
0.50	0.25	0.0	0.466537	2.862167	1.431084	0.000000
		0.1	0.302095	2.730081	1.365040	−3.242957
		0.2	−0.165484	2.389084	1.194542	−5.998958
		0.5	−2.744036	1.185185	0.592593	−10.331485
		1.0	−8.404084	0.332522	0.166261	−11.854431
		5.0	−57.987991	0.003926	0.001963	−12.527109

Acknowledgements

Thanks are due to Professor D.K. Yue of M.I.T. for providing the motivation for this work, and to Professor J.N. Newman of M.I.T. and Professor J. Martin of the University of Edinburgh for helpful discussions.

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